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A five-vertex model interpretation of one-dimensional traffic flow

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Abstract. Here we solve a discrete one-dimensional traffic flow problem by mapping the allowed sets of car trajectories onto a line representation of the five-vertex model configurations. The fundamental flow diagram, obtained previously in a grand canonical ensemble, is rederived. Fluctuations of the flow are described quantitatively and two critical exponents are defined. The zero-density limit is studied by considering an ensemble of single directed self-avoiding loops on a finite torus.

1. Introduction

The modelling and simulation of traffic flow has become a major field of research in theoretical physics in the past years [1]. Beyond the classical approaches, a new class of dicrete cellular automaton models has attracted much interest due to its simplicity and computational efficiency [2]. However, up to now only few exact results are known, even for the simplest cases of next-neighbour hopping [3]. Some new techniques have been developed to reproduce at least the fundamental diagrams (flow against density) in the stationary state [4]. Therefore, new approaches to this problem are highly recommended. The subject of this paper will be a new interpretation in terms of the five-vertex model.

Recently [5] an exactly solved statistical mechanical model of one-dimensional traffic flow has been constructed by using a one-to-one mapping of the dimer configurations on a hexagonal lattice into the set of directed self-avoiding walks (DSAWs) on a square lattice. The latter were interpreted as trajectories of cars in discrete space and time by identifying the vertical direction with the temporal axis and the horizontal direction with the spatial axis. This mapping allowed the authors to use the exact solution of the dimer model due to Kasteleyn [6] for calculation of the fundamental traffic flow diagram and the car–car correlation functions. However, the method intrinsically involves the grand canonical ensemble with respect to the number of cars (trajectories). In the thermodynamic limit their density ρ is controlled by two independent parameters, namely the properly chosen statistical weights x and t of dimers on lattice edges with different orientation. Note that the idea for one-to-one mapping of the allowed dimer configurations on a hexagonal lattice into non-intersecting path configurations on the square lattice has been used by Garrod [7] in his treatment of a stochastic model of three-dimensional crystal growth.

Here we propose a solution of the same one-dimensional traffic flow problem [5] but in the canonical ensemble with fixed number of cars N_c. The solution is obtained by mapping the allowed sets of car trajectories on a square lattice onto a line representation of the five-vertex model configurations, see [8]. We identify each spatial step of a car trajectory with a right arrow, and each temporal step with a down arrow. The exclusion of intersecting trajectories is achieved then by setting to zero the weight of vertex type 3 in the standard notation [8]. Since, for a given lattice with periodic boundary conditions, the number of trajectories fixes the total number of time steps, the weight t of a time step becomes redundant (in the following we set $t = 1$), and the remaining independent parameters are the car density ρ and the weight x of a spatial step (of vertex type 1). Then the problem of finding the average flow $q = q(\rho, x)$ can be formulated in the language of the five-vertex model as a problem of finding the average horizontal polarization, $p_h = 2q - 1$, in a subspace of fixed vertical polarization, $p_v = 1 - 2\rho$, for the one-parameter family of vertex weights given by

$$\omega_1 = x \qquad \omega_2 = \omega_4 = 1 \qquad \omega_3 = 0 \qquad \omega_5 = \omega_6 = x^{1/2}. \tag{1}$$

One readily sees that our model belongs to the class of free-fermion models [8].

The fact that the five-vertex model is equivalent to a problem of many non-intersecting directed self-avoiding walks is known and widely used in various physical problems, see for example [9, 10] and references therein. Note that in most applications the five-vertex model is considered in the grand canonical ensemble and the treatment is confined to some subspaces of parameters which are not always suitable for our problem. Thus, in [9] the model is confined to the one-dimensional subspace of independent parameters defined by the weights $\omega_1 = 1$, $\omega_2 = 0$, and $\omega_3 = \omega_4 = \omega_5 = \omega_6 = \exp(-\beta\epsilon)$. Taking into account the opposite assignment of lines to horizontal arrows in our case, we see that the above family of weights contains (at infinite temperature, $\beta = 0$) only the point $x = 1$ of our model. The analysis of the phase diagram of the five-vertex model in the space of two independent parameters given in [11] is inappropriate again, since the free-field condition assumed there implies equal weights of the steps in space and time ($x = t = 1$). The most general five-vertex model with three independent parameters has been studied by Noh and Kim [10] in the context of interacting domain walls. Our one-dimensional subspace corresponds to setting $x_2 = y_1 = 1$ in their parametrization of the vertex weights.

In section 2 we make precise the isomorphism between the car trajectories and the line configurations of the five-vertex model on a finite torus. Then, from the results of [10] we deduce the exact expressions for the density of the configurational free energy $\kappa(\rho, x)$ and the flow $q(\rho, x)$ in the thermodynamic limit. The flow diagram coincides with the one obtained in [5]. In section 3 we study the fluctuations in the flow and their relation to the features of the fundamental flow diagram. To give a simple picture of the singularities emerging in the zero-density limit, in section 4 we consider the case of just one car which maps on a single DSAW.

2. The model

First we have to make precise our traffic interpretation of the lines of down- and right-arrows in the five-vertex model configurations on a finite torus. Ambiguity may arise from the fact that these lines form closed loops which wind around the torus in space and/or time directions. Accordingly, the configurations can be classified by the winding numbers ω_s and ω_t, respectively, which are the same for all closed lines in a given configuration. The allowed sets $\{\omega_s, \omega_t\}$ of winding numbers are of the following types. Obviously, $\{0, 0\}$ is the empty lattice. The configurations of the class $\{0, 1\}$ contain lines of down-arrows only, winding once in the time direction, and, therefore, can be interpreted as trajectories of stopped cars. The class $\{1, 0\}$ contains lines of right-arrows only, winding once in the space direction. Although these configurations have no meaningful interpretation in terms of car trajectories, they are of no relevance to the present work, since we consider the subspace of configurations with a fixed non-zero number N_c of down-arrows. The remaining classes of configurations are of the types $\{\omega_s, 1\}$, with $\omega_s = 1, \ldots, \omega_s^{\max}$, and $\{1, \omega_t\}$, with $\omega_t = 1, \ldots, \omega_t^{\max}$. For the sake of concreteness, we consider a torus consisting of L columns and M rows of sites. Choose the origin of a coordinate system at an arbitrary site and label it by $(1, 1)$. Let then (p, q) be the site in column p and row q, $p = 1, \ldots, L$, $q = 1, \ldots, M$. The number of cars N_c, by definition, equals the number of down-arrows in each row of vertical edges connecting sites (p, q) with sites $(p, q+1)$ for $p = 1, \ldots, L$ and any fixed q. Obviously, all cars in a configuration C of type $\{\omega_s, 1\}$ pass the same distance $\omega_s L$ for time M, i.e. their average velocity is

$$v(C \in \{\omega_s, 1\}) = \frac{\omega_s L}{M}. \tag{2}$$

On the other hand, in configurations of the type $\{1, \omega_t\}$ each winding of a closed line along the periodic time direction is interpreted as a separate car trajectory associated with the corresponding down-arrow in any fixed row of vertical edges. Thus the number of cars is a multiple of ω_t, and for each group of ω_t cars, generated by one closed line, the total sum of the distances passed by the individual cars is L. Therefore, the average velocity of a car in such a configuration is

$$v(C \in \{1, \omega_t\}) = \frac{L}{\omega_t M}. \tag{3}$$

Clearly, a shortcoming of this interpretation is the introduction of spurious correlations between end points and starting points of different car trajectories belonging to successive turns of one closed line. However, these are expected to have no effect in the thermodynamic limit.

Next we note that the winding numbers ω_s, ω_t and the total numbers of occupied horizontal, N_x, and vertical, N_t, edges in any configuration C with $\omega_t(C) \geqslant 1$ are related by the equations

$$\begin{aligned} N_x(C) &= \rho(C) L^2 \omega_s(C)/\omega_t(C) \\ N_t(C) &= \rho(C) LM \end{aligned} \tag{4}$$

where $\rho(C) = N_c(C)/L$ is the density of cars (down-arrows in a row of vertical edges). Therefore, from equations (2)–(4) we conclude that the average velocity $v(C)$ in any configuration with $\omega_t(C) \geqslant 1$ is given by

$$v(C) = \frac{N_x(C)}{N_t(C)} = \frac{N_x(C)}{\rho(C) LM}. \tag{5}$$

Now we confine our consideration to the set of configurations C_ρ with a fixed number of cars, $N_c(C_\rho) = \rho L \geqslant 1$. The partition function of the model is given by

$$Z_{L,M}(\rho, x) = \sum_{C_\rho} x^{N_x(C_\rho)} \tag{6}$$

and the configurational free energy density is defined as

$$\kappa_{L,M}(\rho, x) = (ML)^{-1} \ln Z_{L,M}(\rho, x). \tag{7}$$

From equations (5)–(7) and the definition of the flow density, $q(C_\rho) := \rho v(C_\rho)$, it follows that the average flow density is

$$q_{L,M}(\rho, x) = \frac{1}{LM} \langle N_x(C_\rho) \rangle = x \frac{\partial}{\partial x} \kappa_{L,M}(\rho, x). \tag{8}$$

The explicit expression for the configurational free energy density (7) in the thermodynamic limit $L \to \infty$, $M \to \infty$ follows from equation (29) of [10], by taking into account the fact that in the free-fermion case the solutions $\{z_j\}$ of the Bethe ansatz equations giving the maximum eigenvalue of the transfer matrix are $z_j = \exp(\mathrm{i}k_j)$, with $k_j = (2j-1)\pi/L$, $j = -N_c/2+1, \ldots, 0, \ldots, N_c/2$ for N_c even, and $k_j = 2j\pi/L$, $j = -(N_c-1)/2, \ldots, 0, \ldots, (N_c-1)/2$ for N_c odd. Note also that their domain-wall density q corresponds to our $1-\rho$ and set their parameter $a = x$. The result is

$$\kappa(\rho, x) = \frac{1}{2\pi}\int_{\pi\rho}^{\pi} \mathrm{d}\phi \ln(1 - 2x\cos\phi + x^2). \tag{9}$$

Hence, the fundamental traffic flow–density relationship which follows from (8) in the thermodynamic limit is

$$q(\rho, x) = \frac{1}{2}(1-\rho) + \frac{1}{2}\,\mathrm{sign}(x-1)\left[1 - \frac{2}{\pi}\arctan\left(\frac{(x+1)}{|x-1|}\tan(\pi\rho/2)\right)\right]. \tag{10}$$

This equation coincides with equation (17) in [5] which has been obtained in the grand canonical ensemble by neglecting the fluctuations in the number of cars. The variable x in it controls the average velocity v_1 of a single car, since for configurations with only one trajectory $v_1 = x$.

Note that for fixed $0 < \rho < 1$ the flow $q(\rho, x)$ monotonically increases with $x \geqslant 0$ from the value $q(\rho, 0) = 0$ to $q(\rho, \infty) = 1-\rho$. On the other hand, the behaviour of the flow as a function of the density ρ has two qualitatively different regimes. When $x < 1$ the flow initially increases with ρ from $q(0, x) = 0$ to the maximum value $q(\rho_{\max}, x) = \frac{1}{2} - \rho_{\max}$, reached at $\rho_{\max} = \pi^{-1}\arccos x$, then decreases to zero at $\rho = 1$. When $x > 1$ the flow monotonically decreases from $q(0, x) = 1$ to $q(1, x) = 0$. The existence of these two regimes has found no explanation in [5]. To shed more light on this fact, in the next section we consider the fluctuations of the flow.

3. Fluctuations of the flow

Obviously, in the ensemble of configurations C_ρ with a fixed density of cars on a finite torus, the fluctuations in the flow $q(C_\rho) = \rho v(C_\rho)$ arise due to fluctuations in the ratio $\omega_s(C)/\omega_t(C)$ of the winding numbers, see equations (4) and (5). The variance of the flow is given by the susceptibility

$$\chi(\rho, x) = \lim_{L\to\infty, M\to\infty}(LM)^{-1}[\langle N_x^2(C_\rho)\rangle - \langle N_x(C_\rho)\rangle^2] = x\frac{\partial}{\partial x}q(\rho, x). \tag{11}$$

By explicit differentiation of equation (10) one obtains

$$\chi(\rho, x) = \frac{2}{\pi}\frac{x\tan(\pi\rho/2)}{(x-1)^2 + (x+1)^2\tan^2(\pi\rho/2)}. \tag{12}$$

It is straightforward to show that $\chi(\rho, x)$, for any fixed $0 < \rho < 1$, grows with $x \in [0, 1)$, reaches maximum at $x = 1$,

$$\max_x \chi(\rho, x) = \chi(\rho, 1) = \frac{1}{2\pi}\cot(\pi\rho/2) \tag{13}$$

and then decreases asymptotically as x^{-2} when $x \to \infty$. Thus, there are infinite fluctuations in the flow only at $x = 1$ in the limit $\rho \downarrow 0$, when the susceptibility diverges as $\pi^{-2}\rho^{-1}$. Hence we derive that the critical exponent with respect to ρ at $\rho_c = 0$ is $\gamma_\rho = 1$. Note that for any $x \neq 1$ the susceptibility vanishes in the limit $\rho \downarrow 0$. In the other limit, $\rho \uparrow 1$, the susceptibility vanishes for all x.

Let us consider in more detail the limit $\rho \downarrow 0$. By expanding the integral in the right-hand side of equation (9) in powers of ρ one easily obtains the expansion for the configurational free energy density. Hence a term-by-term differentiation yields the average flow. The corresponding analytical expressions depend on weather x is less or greater than unity:

$$\kappa(\rho, x) = -\rho \ln(1-x) + \mathrm{O}(\rho^3) \qquad q(\rho, x) = \rho \frac{x}{1-x} + \mathrm{O}(\rho^3) \tag{14}$$

when $\rho \downarrow 0, x < 1$, and

$$\kappa(\rho, x) = \ln x - \rho \ln(x-1) + \mathrm{O}(\rho^3) \qquad q(\rho, x) = 1 - \rho \frac{x}{x-1} + \mathrm{O}(\rho^3) \tag{15}$$

when $\rho \downarrow 0, x > 1$. For any $x \neq 1$ the susceptibility is given by

$$\chi(\rho, x) = \frac{\rho x}{(x-1)^2} + \mathrm{O}(\rho^3) \qquad \rho \downarrow 0, x \neq 1. \tag{16}$$

The above equation defines a critical exponent with respect to x at $x_c = 1$, namely $\gamma_x = 2$.

By using the same method, in the limit $\rho \uparrow 1$ we obtain for all x,

$$\begin{aligned} &\kappa(\rho, x) = (1-\rho)\ln(x+1) + \mathrm{O}[(1-\rho)^3] \\ &q(\rho, x) \simeq (1-\rho)\frac{x}{x+1} \qquad \chi(\rho, x) \simeq (1-\rho)\frac{x}{(x+1)^2}. \end{aligned} \tag{17}$$

4. The zero-density limit

The origin of the singularities in the zero-density limit can be easily understood in the case of a single car. According to the definition of our model on a finite torus, a fixed value $N_c(C) = 1$ restricts the set of configurations of the five-vertex model to the subset $\{\omega, 1\}_1$ which contains exactly one closed line of down- and right-arrows winding $\omega \geqslant 0$ times in the spatial direction and just once in the temporal one. Equivalently, we have an ensemble of single closed DSAWs on a $L \times M$ torus with partition function

$$Z_{L,M}(L^{-1}, x) = L \sum_{\omega=0}^{[M(L-1)/L]} a_{L,M}(\omega L) x^{\omega L}. \tag{18}$$

Here $a_{L,M}(\omega L)$ is the number of single closed DSAWs which pass through the origin $(1, 1)$ and have exactly M vertical steps and ωL horizontal steps. The factor L in the right-hand side is due to the fact that a walk can pass through any of the L sites in the first row. The numbers $a_{L,M}(\omega L)$ have a simple combinatorial meaning: they enumerate the ways in which the ωL horizontal steps can be distributed among the M rows in such a way that in each row there are zero or no more than $L-1$ horizontal steps (to avoid self-intersection). The generating function of these numbers is [12]:

$$\left(\sum_{m=0}^{L-1} t^m\right)^M = \sum_{k=0}^{(L-1)M} a_{L,M}(k) t^k. \tag{19}$$

On the other hand, the left-hand side of the above equation equals

$$(1-t^L)^M (1-t)^{-M} = \sum_{k=0}^{\infty} \sum_{m=0}^{M} (-1)^m \binom{M}{m} \binom{M-1+k}{M-1} t^{k+mL}. \tag{20}$$

Hence we obtain

$$a_{L,M}(\omega L) = \sum_{n=0}^{\omega} (-1)^n \binom{M}{n} \binom{M-1+(\omega-n)L}{M-1}. \tag{21}$$

The symmetry and unimodality properties of the series of positive integers $a_{L,M}(k)$, $k = 0, \ldots, (L-1)M$, follow from (19) and the repeated application of theorem 3.9 of [13] (see also theorem 4.2 therein):

$$a_{L,M}(k) = a_{L,M}((L-1)M - k) \qquad k = 0, \ldots, (L-1)M \tag{22}$$

$$a_{L,M}(k+1) \geqslant a_{L,M}(k) \qquad k = 0, \ldots, [(L-1)M/2] - 1. \tag{23}$$

Note that $a_{L,M}(0) = a_{L,M}((L-1)M) = 1$ yields the lower bound on $a_{L,M}(k)$; a simple upper bound follows by setting $t = 1$ in (19), hence

$$1 \leqslant a_{L,M}(k) \leqslant L^M \qquad k = 0, \ldots, (L-1)M. \tag{24}$$

The above bounds suffice to prove that the thermodynamic limit for the configurational free energy density (7) at $\rho = L^{-1}$ exists and equals

$$\kappa(0, x) = \lim_{L\to\infty, M\to\infty} \kappa_{L,M}(L^{-1}, x) = \begin{cases} 0 & \text{if } 0 \leqslant x < 1 \\ \ln x & \text{if } x > 1. \end{cases} \tag{25}$$

In order to obtain the behaviour of the flow density, see equations (7) and (8), we set $x = \mathrm{e}^y$ and make use of the fact that $\kappa_{L,M}(L^{-1}, \mathrm{e}^y)$, $L, M = 2, 3, \ldots$, is a sequence of convex functions of $y \in (-\infty, \infty)$. Indeed, for the first and second derivatives with respect to y one obtains

$$q_{L,M}(L^{-1}, \mathrm{e}^y) := \frac{\partial}{\partial y}\kappa_{L,M}(L^{-1}, \mathrm{e}^y) = \frac{1}{M}\langle\omega\rangle \tag{26}$$

$$\chi_{L,M}(L^{-1}, \mathrm{e}^y) := \frac{\partial^2}{\partial y^2}\kappa_{L,M}(L^{-1}, \mathrm{e}^y) = \frac{L}{M}\langle(\omega - \langle\omega\rangle)^2\rangle \geqslant 0. \tag{27}$$

Obviously, the above sequence of convex functions converges to the convex function of $y \in (-\infty, \infty)$ given by the right-hand side of (25) after the substitution $x = \mathrm{e}^y$. Therefore, from the Griffiths–Fisher lemma [14, 15] on the sequence of first derivatives of convex functions it follows that the flow density in the thermodynamic limit is given by the step-like function

$$q(0, x) = \lim_{L\to\infty, M\to\infty} x\frac{\partial}{\partial x}\kappa_{L,M}(L^{-1}, x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1. \end{cases} \tag{28}$$

Note that equations (25) and (28) reproduce correctly the $\rho \downarrow 0$ limit of equations (14) and (15).

Next we show that the value of the flow density at $x = 1$ is a direct consequence of the symmetry property (23), provided $M = pL$, where p is an arbitrary positive integer. Indeed, in this case from (18) and (23) it follows that

$$Z_{L,pL}(L^{-1}, x) = x^{pL(L-1)} Z_{L,pL}(L^{-1}, x^{-1}). \tag{29}$$

By differentiating the above identity with respect to x and setting then $x = 1$ we obtain for the value of the first derivative of the partition function at that point

$$Z'_{L,pL}(L^{-1}, 1) = \tfrac{1}{2} pL(L-1) Z_{L,pL}(L^{-1}, 1). \tag{30}$$

Hence, the finite-size flow density, see equations (7) and (8), at $x = 1$ equals

$$q_{L,pL}(L^{-1}, 1) = \tfrac{1}{2}(1 - L^{-1}). \tag{31}$$

From these results one may conclude that at $x = 1$ the partition function (18) is dominated by the terms with $\omega \simeq \frac{1}{2}M$, which are the largest ones in view of the symmetry and unimodality properties (23) and (23). When $x < 1$ the representative configurations have winding numbers ω which are of order smaller than M, while for $x > 1$ the difference of ω from the maximum number of windings is of order smaller than M itself. Thus, even a single car provides a unit flow at $x > 1$ by filling the torus with almost close-packed trajectory (a car moving with 'infinite' velocity). Naturally, large fluctuations are expected to occur exactly at $x = 1$, where $\chi_{L,M}(L^{-1}, 1)$ may diverge as $L, M \to \infty$.

5. Discussion

The exact solution of the five-vertex model in sectors of fixed density of down-arrows in a row is mapped onto a one-dimensional traffic flow problem. The model represents an equilibrium statistical mechanical analogue in $(1 + 1)$-dimensional space of the modified Nagel–Schreckenberg cellular automaton with $v_{\max} = \infty$, suggested and solved for the steady-state properties under periodic boundary conditions in [5]. The coincidence of the obtained traffic flow–density relationship with the one derived previously in the grand canonical ensemble [5] shows that the fluctuations of the car density are unessential for the fundamental diagram within the model.

An exact explicit expression is derived for the variance of the flow density in the thermodynamic limit, and its singularities are described in terms of critical exponents. It is shown that the two qualitatively different regimes of the flow density as a function of the car density are separated by a line of maximum flow fluctuations. The zero-density limit is studied by considering an ensemble of single closed DSAWs on a finite torus. It is proved that in the thermodynamic limit of zero car density the flow density becomes a step-like function of the single car velocity. The unphysical solution which yields unit flow density at zero car density is explained as arising from a few cars moving with infinite velocity.

Some comments are in place here on the relation of our results to those of the recent paper by Sasvári and Kertész [16], where the singularity of the Nagel–Schreckenberg model [2] in the zero-density $v_{\max} = \infty$ limit has been considered. First, we emphasize that in the present model statistical weights are assigned to trajectories as a whole and, therefore, its exact interpretation in terms of a cellular automaton with local rules seems hardly feasible. As already mentioned, the closest analogue known is the $v_{\max} = \infty$ stochastic cellular automaton suggested in [5], which, however, differs essentially from the $v_{\max} = \infty$ limit of the Nagel–Schreckenberg model in the aspect that the former has no velocity memory. The lack of velocity memory is a great over-simplification with regard to vehicular traffic modelling, since no rules for individual driving behaviour are taken into account, and very large velocity fluctuations are allowed to occur. However, the comparison of the fundamental flow-density diagram for the present model, see figure 2 in [5], with that obtained by computer simulations on the $v_{\max} = \infty$ limit of the Nagel–Schreckenberg model, see figure 12 in [16], reveals some striking similarities. In the latter case the graph of the flow $j(\rho)$, as a function of the density ρ, seems composed of two different branches of the flow $q(\rho, x)$ in the former case. Namely, for very small densities ρ the flow $j(\rho)$ behaves like the 'unphysical' branch of $q(\rho, x)$ at some $x = 1 + \epsilon$, $0 < \epsilon \ll 1$, since it tends to unity in the zero-density limit, and decreases quickly with the increase of ρ up to the value ρ_t. At the crossover point $\rho = \rho_t$, which was found [16] to decrease with the system size L as $\rho_t \simeq L^{-t}$, $t \cong 0.5$, the regime changes drastically to what the authors of [16] call a plateau, and what we find to resemble very much the 'physical' branch of $q(\rho, x)$ at some $x = 1 - \epsilon$, $0 < \epsilon \ll 1$. This observation, together with the fact that the point $\rho = \rho_t$ lies in the region of anomalously high fluctuations of the flow, suggests the speculation that the crossover takes place between a metastable and a stable branch of the flow in the $v_{\max} = \infty$ Nagel–Schreckenberg model. Obviously, the above conjecture is based on the comparison of the fundamental diagrams of two different (although similar in some sense) models and, hence, its validity needs further investigation.

Finally, we mention that the cellular automaton, with $v_{\max} = \infty$ and without velocity memory [5], has found a recent application in modelling the 'bottleneck' situation in traffic [17]. To this end open boundary conditions have been used, the cars being injected into the system with a prescribed rate at the left boundary, and leave it freely through the right one.

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